

Supplementary online material to  
“Information tradeoffs in dynamic financial markets”

Efstathios Avdis\*  
*University of Alberta, Canada*

## 1. The value of information in continuous time

In this document I address how the information-market equilibrium works in an economy with more than two periods by deriving the value of information in the limiting case of continuous time and infinite horizon. This is meant as a robustness check to show that complementarities obtain even when the model features interim dividend payouts, interim consumption and a long horizon. A further motivation for this derivation is to connect with existing literature, namely the asset-pricing study of exogenous asymmetric information of Wang (1993). I use that model for the financial market and derive the corresponding equilibrium in the information market.

The economy is made up of a continuum of ex-ante identical investors of total mass one. Every investor has constant-absolute-risk-aversion (CARA) preferences with coefficient  $\delta$ . Everyone can invest in a safe bond with constant interest rate  $r$  and in a risky stock the dividend process of which is

$$dD_t = \phi_D(\mu_t - D_t)dt + \sigma_D dB_t^D, \quad (1)$$

where  $B_t^D$  is a Brownian motion driving the dividends. The growth rate  $\mu_t$  is not freely observable, but it is known that it follows the process

$$d\mu_t = \phi_\mu(m - \mu_t)dt + \sigma_\mu dB_t^\mu, \quad (2)$$

---

\*University of Alberta, 3-30B Business Building, Edmonton, Alberta, Canada T6G 2R6, email: avdis@ualberta.ca

where  $B_t^\mu$  is a Brownian motion independent of  $B_t^D$ . In the financial market there are  $\lambda$  informed agents that observe the full history of  $\mu_t$  and  $1 - \lambda$  uninformed agents that rely only on prices and dividends to infer as well as possible the value of  $\mu_t$ . The total stock supply at time  $t$  is  $1 + \theta_t$ , where  $\theta_t$  is not observable,

$$d\theta_t = -\phi_\theta \theta_t dt + \sigma_\theta dB_t^\theta, \quad (3)$$

and  $B_t^\theta$  is a Brownian motion independent of  $B_t^\mu$  and  $B_t^D$ . In this model Wang (1993) studies the asset pricing implications of exogenous information asymmetry. I endogenize the fraction  $\lambda$  of informed agents as a function of information acquisition costs. I begin with a brief description of the equilibrium in the financial market.

The equilibrium price process is

$$P_t = p_0 + p_D D_t + p_\mu \mu_t + p_\theta \theta_t + p_{\hat{\mu}} \hat{\mu}_t + p_{\hat{\theta}} \hat{\theta}_t. \quad (4)$$

The information that the informed have is the complete history of the dividend  $D_t$ , of the price  $P_t$  and of the dividend growth rate  $\mu_t$ . Let the  $\sigma$ -algebra  $\mathcal{F}_t^i$  represent this information. The information  $\mathcal{F}_t^u$  that the uninformed have at time  $t$  is the complete history of the dividend  $D_t$  and the price  $P_t$  only. The inferences  $\hat{\mu}_t$  and  $\hat{\theta}_t$  are the best estimates of  $\mu_t$  and  $\theta_t$  given the information  $\mathcal{F}_t^u$ ,

$$\hat{\mu}_t = \mathbb{E} \left[ \mu_t \middle| \mathcal{F}_t^u \right] = \mathbb{E} \left[ \mu_t \middle| \{D_s, P_s\}_{0 \leq s \leq t} \right], \quad (5)$$

and

$$\hat{\theta}_t = \mathbb{E} \left[ \theta_t \middle| \mathcal{F}_t^u \right] = \mathbb{E} \left[ \theta_t \middle| \{D_s, P_s\}_{0 \leq s \leq t} \right]. \quad (6)$$

These estimates are given by Kalman-Bucy filtering, with steady-state solution

$$d \begin{pmatrix} \hat{\mu}_t \\ \hat{\theta}_t \end{pmatrix} = \begin{pmatrix} \phi_\mu (m - \hat{\mu}_t) \\ -\phi_\theta \hat{\theta}_t \end{pmatrix} dt + h(q_{xx})^{\frac{1}{2}} d\hat{B}_t. \quad (7)$$

The filtering innovation  $\hat{B}_t$  is a two-dimensional vector Brownian motion and the matrices  $q_{xx}$  and  $h$  are constants provided in terms of the model parameters in Appendix A.

The demand of each agent group is the solution to a portfolio problem taking the returns in

excess of the risk-free rate,  $R_t$ , as given. The excess returns follow the process

$$dR_t = (D_t - rP_t)dt + dP_t. \quad (8)$$

As Wang (1993) shows the state vector for the uninformed can be reduced to the column vector  $S_t^u = (1 \ \hat{\theta}_t)^T$  and the state vector for the informed can be reduced to the column vector  $S_t^i = (1 \ \theta_t \ \hat{\mu}_t - \mu_t)^T$ . This reduction also gives explicit expressions for  $p_D$  and  $p_\mu + p_{\hat{\mu}}$ . Consider now the portfolio problem of an investor in group  $j$  for  $j = i, u$ . At time  $t$  the dollar amount invested in the stock is  $X_t^j$ , the wealth is  $W_t^j$  and consumption is  $c_t^j$ . Let  $\nu$  be the discount factor. The portfolio selection problem is

$$\begin{aligned} \max_{\{c_s^j, X_s^j\}_{t \leq s \leq \infty}} \quad & \mathbb{E} \left[ \int_t^\infty e^{-\nu s} \left( -e^{-\delta c_s^j} \right) ds \middle| \mathcal{F}_t^j \right] \\ \text{s.t.} \quad & dW_t^j = \left( rW_t^j - c_t^j \right) dt + X_t^j dR_t \\ & dR_t = m_R^j S_t^j dt + v_R^j d\hat{B}_t \\ & dS_t^j = m_S^j S_t^j dt + v_S^j d\hat{B}_t, \end{aligned} \quad (9)$$

where  $m_S^j$ ,  $v_S^j$ ,  $m_R^j$  and  $v_R^j$  are constant matrices. It is well known that the value function for this type of problem is separable in time, state variables and wealth. It has the form  $e^{-\nu t} J^j(W_t^j, S_t^j)$  where

$$J^j(W_t^j, S_t^j) = -A^j e^{-r\delta W_t^j - \frac{1}{2} S_t^{jT} \gamma^j S_t^j}. \quad (10)$$

The constant  $A^j$  is scalar and  $\gamma^j$  is a matrix of constants characterized in Appendix B.

The optimal demand  $X_t^{j*}$  is linear in the state vector of agent group  $j$ , that is,  $X_t^{j*} = d^j S_t^j$  where  $d^j$  is a row vector that depends on the coefficients of the price process and the parameters of the economy. I give  $d^i$  and  $d^u$  in Appendix B. The stock market clears when the aggregate investor demand equals the noisy supply,

$$\lambda X_t^{i*} + (1 - \lambda) X_t^{u*} = 1 + \theta_t. \quad (11)$$

Matching coefficients in the underlying state variables of the two agent groups gives three non-linear equations. These pin down the coefficients in the price function in terms of  $\lambda$ .

### 1.1. The equilibrium in the information market

Next I endogenize the fraction of informed agents by using the same equilibrium concept as in definition ?? of the main text, where the financial equilibrium is now a sequence of prices  $\{P_t(\lambda)\}_{0 \leq t \leq \infty}$  given by the financial market equilibrium of Wang (1993) that I have just described. Each agent has the option to subscribe to the full observations of  $\mu_t$  at  $t = 0$  by incurring the cost  $\kappa_0$ . After  $t = 0$  the agents cannot change their information status. The equilibrium number of informed agents  $\lambda^*$  is determined in the same way as in discrete time, so here I focus on describing the derivation of the value of information. The value of information  $\Psi_0(\lambda)$  is defined as

$$J^u(W_0, S_0^u; \lambda) = \mathbb{E} \left[ J^i(W_0 - \Psi_0(\lambda), S_0^i; \lambda) \middle| \mathcal{F}_0^u \right]. \quad (12)$$

The calculation of  $\Psi_0(\lambda)$  is in closed form in terms of the solution of the portfolio problems and the price coefficients. Because, however, the price coefficients have to be solved numerically, I do not have a completely closed form of  $\Psi_0(\lambda)$  in terms of the model parameters. Similarly to section ?? of the main text, the value of information can be written as

$$\Psi_0(\lambda) = \frac{1}{r\delta} \log \left( \frac{J^u(W_0, S_0^u; \lambda)}{\mathbb{E} [J^i(W_0, S_0^i; \lambda) | \mathcal{F}_0^u]} \right). \quad (13)$$

The result of this calculation depends on the uninformed agents' prior supply estimate,  $\hat{\theta}_0$ . I set  $\hat{\theta}_0$  equal to the long-run mean of the  $\theta$  process, which is zero. This is the same as requiring that the uninformed prior is specified correctly. In Fig. 1 I show the value of information  $\Psi_0(\lambda)$  and the associated asymptotic inference qualities of dividend forecast and supply. As more informed agents enter the economy prices become more informative about dividends but less informative about supply, as witnessed by that  $AVar(\mu_t | \mathcal{F}_t^u)$  decreases in  $\lambda$  and that  $AVar(\theta_t | \mathcal{F}_t^u)$  increases in  $\lambda$ . The question now is, when do complementarities arise? As the discussion above shows, complementarities in information acquisition arise when the persistence  $\rho$  of the supply is low. A calculation based on matching the correlation structure of an AR(1) process with that of the Ornstein-Uhlenbeck process shows that  $\rho = e^{-\phi_\theta \Delta t}$ , where  $\Delta t$  is the time between consecutive periods of realizations of the AR(1) process. Therefore complementarities should arise when  $\phi_\theta$  is high. As the top plot of Fig. 1 verifies, when  $\phi_\theta$  is high, that  $AVar(\theta_t | \mathcal{F}_t^u)$  is increasing in  $\lambda$  makes

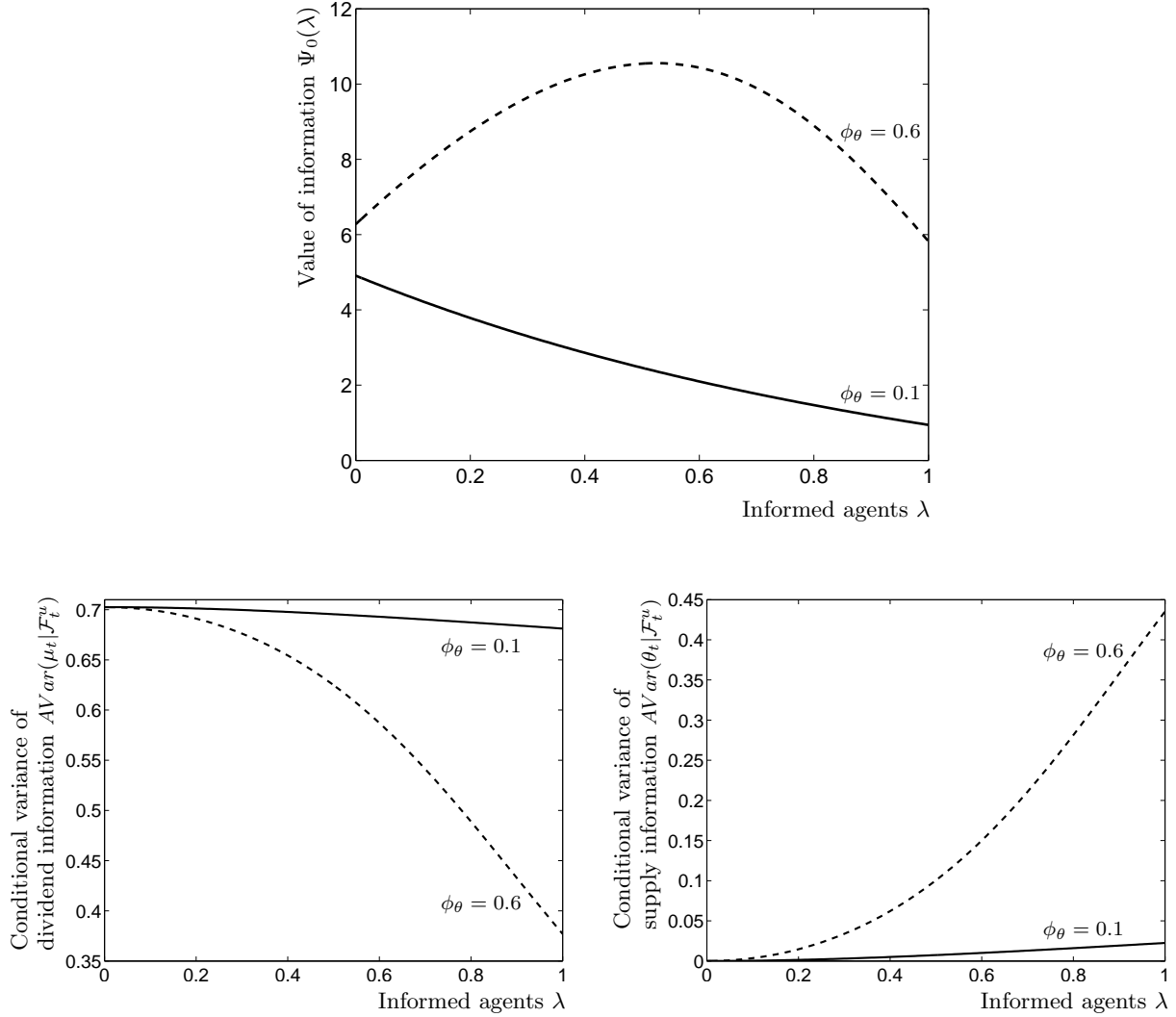


Fig. 1. The value of information  $\Psi_0(\lambda)$ , the asymptotic conditional variance of dividend information  $AVar(\mu_t|\mathcal{F}_t^u)$ , and the asymptotic conditional variance of supply  $AVar(\theta_t|\mathcal{F}_t^u)$ , as a function of informed agents  $\lambda$ . The informed agents observe the levels of dividend information  $\mu_t$  and supply  $\theta_t$  at time  $t$ , whereas the information set  $\mathcal{F}_t^u$  of the uninformed agents contains the history of prices and dividends up to time  $t$ . The CARA coefficient is  $\delta = 3$ , the interest rate is  $r = 0.05$ , and the long-run mean of the dividend process is  $m = 0.8$ . The mean reversion of the dividend process is  $\phi_D = 0.4$  and the mean reversion of the dividend-information process is  $\phi_\mu = 0.2$ . The volatility of the dividend process is  $\sigma_D = 1$ , the volatility of the dividend-information process is  $\sigma_\mu = 0.6$ , and the volatility of the supply process is such that  $\sigma_\theta^2/2\phi_\theta = 1$ . Solid curves for  $\phi_\theta = 0.1$  and dashed curves for  $\phi_\theta = 0.6$ .

the value of information non-monotonic in  $\lambda$ .

Finally let me address the differences between the model in discrete time and the model in continuous time. One advantage of the two-period model is that I can have an expression for the value of information in terms of conditional moments of returns. In continuous time the best available description of the value of information is as a ratio of value functions. But perhaps most importantly, the economic forces of the continuous-time model are clearly expressed already in the two-period model. This says that the length of the economy does not matter for the value of information. To gain some intuition about why this is the case, consider the following argument. Suppose that the economy was in continuous time but that the world ended at a random date  $\tau$ . To keep things simple further suppose that  $\tau$  was determined by the arrival of a Poisson shock of rate  $\nu^\tau$ , independently of everything else in the economy. By a standard exchange-of-integrals argument the objective value function of each agent group  $j$  would be

$$\mathbb{E} \left[ \int_t^\tau e^{-\nu s} \left( -e^{-\delta c_s^j} \right) ds \middle| \mathcal{F}_t^j \right] = \mathbb{E} \left[ \int_t^\infty e^{-(\nu+\nu^\tau)s} \left( -e^{-\delta c_s^j} \right) ds \middle| \mathcal{F}_t^j \right]. \quad (14)$$

That is, the only change in the economy is that the discount factor  $\nu$  has increased by  $\nu^\tau$ . But as we can see in Appendix B, the discount factor washes out completely in the expression for  $\Psi_0(\lambda)$ . Therefore in this context the length of the economy does not matter for the value of information.

## Appendix A. The filtering problem of the uninformed agents

The matrices in the solution of the filtering problem of the uninformed agents are

$$q_{xx} = \begin{bmatrix} p_\theta^2 \sigma_\theta^2 + p_\mu^2 \sigma_\mu^2 & 0 \\ 0 & \sigma_D^2 \end{bmatrix}, \quad (15)$$

and

$$h = \frac{1}{H} \begin{bmatrix} \frac{1}{p_\theta} \frac{p_\mu}{p_\theta} \sigma_\mu^2 (\phi_\theta + G) & \sigma_\theta^2 \frac{\sigma_\mu^2}{\sigma_D^2} \phi_D \\ \frac{1}{p_\theta} \sigma_\theta^2 (\phi_\mu + G) & -\frac{p_\mu}{p_\theta} \sigma_\theta^2 \frac{\sigma_\mu^2}{\sigma_D^2} \phi_D \end{bmatrix}, \quad (16)$$

where

$$H = \left( \frac{p_\mu}{p_\theta} \right)^2 \sigma_\mu^2 (\phi_\theta + G) + \sigma_\theta^2 (\phi_\mu + G), \quad (17)$$

and

$$G = \sqrt{\frac{\phi_\theta^2 \left(\frac{p_\mu}{p_\theta}\right)^2 \sigma_\mu^2 + \sigma_\theta^2 \left(\phi_\mu^2 + \frac{\sigma_\mu^2 \phi_D^2}{\sigma_D^2}\right)}{\sigma_\theta^2 + \left(\frac{p_\mu}{p_\theta}\right)^2 \sigma_\mu^2}}. \quad (18)$$

## Appendix B. Portfolio choice

For  $j = i, u$ , the value function is

$$J^j(W_t^j, S_t^j) = -e^{-r\delta W_t^j - \frac{1}{2}(\alpha^j + S_t^{jT} \gamma^j S_t^j)}. \quad (19)$$

$\alpha^j$  is the scalar

$$\alpha^j = \frac{1}{r} \text{tr} \left( \gamma^j v_S^j v_S^{jT} \right) + 2 \left[ \frac{\nu}{r} + \ln(r) - 1 \right], \quad (20)$$

where  $\gamma^u$  is a  $(2 \times 2)$  matrix and  $\gamma^i$  is a  $(3 \times 3)$  matrix. In particular, for  $j = i, u$ ,  $\gamma^j$  is the solution to the Algebraic Riccati Equation

$$\begin{aligned} 0 = & \gamma \left[ v_R^j v_R^{jT} \left( m_S^j - \frac{r}{2} I \right) - v_S^j v_R^{jT} m_R^j \right] + \left[ v_R^j v_R^{jT} \left( m_S^{jT} - \frac{r}{2} I \right) - m_R^{jT} v_R^j v_S^{jT} \right] \gamma \\ & + \gamma v_S^j \left( v_R^{jT} v_R^j - v_R^j v_R^{jT} I \right) v_S^{jT} \gamma + m_R^{jT} m_R^j. \end{aligned} \quad (21)$$

The optimal demand coefficient for  $j = i, u$  is the matrix

$$d^j = \frac{1}{r\delta v_R^j v_R^{jT}} \left( m_R^j - v_R^j v_S^{jT} \gamma^j \right). \quad (22)$$

For uninformed agents,

$$m_R^u = \left[ (p_\mu + p_{\hat{\mu}}) \phi_\mu m - r p_0 \quad -(r + \phi_\theta)(p_\theta + p_{\hat{\theta}}) \right], \quad (23)$$

$$v_R^u = \left( \left[ 0 \quad p_D \right] + \left[ (p_\mu + p_{\hat{\mu}}) \quad (p_\theta + p_{\hat{\theta}}) \right] h \right) (q_{xx})^{\frac{1}{2}}, \quad (24)$$

$$m_S^u = \begin{bmatrix} 0 & 0 \\ 0 & -\phi_\theta \end{bmatrix}, \quad (25)$$

$$v_S^u = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} h (q_{xx})^{\frac{1}{2}}. \quad (26)$$

For informed agents,

$$m_R^i = \begin{bmatrix} (p_\mu + p_{\hat{\mu}})\phi_\mu m - rp_0 & -r(p_\theta + p_{\hat{\theta}}) & -r\left(p_{\hat{\mu}} - p_\mu \frac{p_{\hat{\theta}}}{p_\theta}\right) \end{bmatrix} \\ + \begin{bmatrix} 0 & (p_\theta + p_{\hat{\theta}}) & \left(p_{\hat{\mu}} - p_\mu \frac{p_{\hat{\theta}}}{p_\theta}\right) \end{bmatrix} m_S^i, \quad (27)$$

$$v_R^i = \begin{bmatrix} 0 & p_D \sigma_D & 0 \end{bmatrix} + (p_\mu + p_{\hat{\mu}}) \begin{bmatrix} 0 & 0 & \sigma_\mu \end{bmatrix} \\ + \begin{bmatrix} 0 & (p_\theta + p_{\hat{\theta}}) & \left(p_{\hat{\mu}} - p_\mu \frac{p_{\hat{\theta}}}{p_\theta}\right) \end{bmatrix} v_S^i, \quad (28)$$

$$m_S^i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\phi_\theta & 0 \\ 0 & 0 & -\phi_e \end{bmatrix}, \quad (29)$$

$$v_S^i = \begin{bmatrix} 0 & 0 & 0 \\ \sigma_\theta & 0 & 0 \\ \frac{1}{H} \frac{p_\mu}{p_\theta} \sigma_\mu^2 \sigma_\theta (\phi_\theta + G) & \frac{1}{H} \sigma_\theta^2 \frac{\sigma_\mu^2 \phi_D}{\sigma_D} & -\frac{1}{H} \sigma_\theta^2 \sigma_\mu (\phi_\mu + G) \end{bmatrix}, \quad (30)$$

where

$$\phi_e = \frac{1}{H} \left\{ \left( \frac{p_\mu}{p_\theta} \right)^2 \sigma_\mu^2 \phi_\theta (\phi_\theta + G) + \sigma_\theta^2 \left[ \frac{\sigma_\mu^2 \phi_D^2}{\sigma_D^2} + \phi_\mu (\phi_\mu + G) \right] \right\}. \quad (31)$$

## References

Wang, J., 1993. A model of intertemporal asset prices under asymmetric information. *The Review of Economic Studies* 60, 249–282.